Level – 2	Daily Tutorial Sheet - 7

91. $I_{m,n} = \int \underset{\text{I}}{\cos^m(x)} \cdot \underset{\text{II}}{\cos(nx)}\, dx$

$I_{m,n} = \cos^m(x)\cdot\frac{\sin(nx)}{n} - \int m\cos^{m-1}(x)(-\sin(x))\left(\frac{\sin(nx)}{n}\right)dx$

$I_{m,n} = \frac{\cos^m(x)\sin(nx)}{n} + \frac{m}{n}\int \cos^{m-1}(x)(\sin(x)\sin(nx))dx$

Now $\cos(nx - x) = \cos(nx)\cos(x) + \sin(nx)\sin(x)$

$\therefore\ \sin(nx)\sin(x) = \cos(n-1)x - \cos(x)\cos(nx)$

$I_{m,n} = \frac{\cos^m(x)\sin(nx)}{n} + \frac{m}{n}\int \cos^{m-1}(x)[\cos(n-1)x - \cos(x)\cos(nx)]dx$

$I_{m,n} = \frac{\cos^m(x)\sin(nx)}{n} - \frac{m}{n}\int \cos^m(x)\cos(nx)dx + \frac{m}{n}\int \cos^{m-1}(x)\cos(n-1)x\ dx$

$I_{m,n}\left[1+\frac{m}{n}\right] = \frac{\cos^m(x)\sin(nx)}{n} + \frac{m}{n} I_{m-1,n-1}$

$I_{m,n} = \frac{\cos^m(x)\sin(nx)}{n} + \frac{m}{m+n} I_{m-1,n-1}$

92. $I_n = \int e^{ax}\sin^n(x)\,dx$

$I_n = \sin^n(x)\int e^{ax}\,dx - \int n\sin^{n-1}(x)\cos(x)\cdot\frac{e^{ax}}{a}dx$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a}\int (\sin^{n-1}(x)\cos(x))\,e^{ax}dx$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a}\left(\sin^{n-1}(x)\cos(x)\right)\int e^{ax} - \frac{n}{a}\int\left[\frac{d}{dx}\left(\sin^{n-1}(x)\cos(x)\right)\int e^{ax}dx\right]dx$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a}\sin^{n-1}(x)\cos(x)\frac{e^{ax}}{a} - \frac{n}{a}\int\left[(n-1)\sin^{n-2}(x)\cos^2(x) - \sin^{n-1}(x)\cdot\sin(x)\right]\frac{e^{ax}}{a}dx$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a^2}e^{ax}\sin^{n-1}(x)\cos(x) - \frac{n(n-1)}{a^2}\int e^{ax}\sin^{n-2}(x)(1-\sin^2(x))dx - \frac{n}{a^2}\int e^{ax}\sin^n(x)\,dx$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a^2}e^{ax}\sin^{n-1}(x)\cos(x) - \frac{n(n-1)}{a^2}\left[I_{n-2} - I_n\right] - \frac{n}{a^2}I_n$

$I_n = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a^2}e^{ax}\sin^{n-1}(x)\cos(x) + I_n\left[\frac{n(n-1)}{a^2} - \frac{n}{a^2}\right] - I_{n-2}\frac{n(n-1)}{a^2}$

$I_n = \left(1 - \frac{n^2-2n}{a^2}\right) = \frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a^2}e^{ax}\sin^{n-1}(x)\cos(x) - I_{n-2}\frac{n(n-1)}{a^2}$

$I_n = \left(\frac{a^2}{a^2-n^2-2n}\right)\frac{1}{a}e^{ax}\sin^n(x) + \frac{n}{a^2}e^{ax}\sin^{n-1}(x)\cos(x) - I_{n-2}\frac{n(n-1)}{a^2}$

93. Substitute $a^2 - x^2 = t^2 \Rightarrow I = -\int\frac{t^2dt}{2a^2-t^2} = \int\left(1-\frac{2a^2}{2a^2-t^2}\right)dt = t + 2a^2\int\frac{dt}{t^2-\left(\sqrt{2}a\right)^2}$

94. Let $\log x = t \Rightarrow \frac{1}{x}dx = dt \Rightarrow I = \int\left(\log t + \frac{1}{t^2}\right)e^t dt = \int\left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right)e^t dt$

$$\Rightarrow \quad I = \int e^t\left(\log t + \frac{1}{t}\right)dt + \int e^t\left(-\frac{1}{t} + \frac{1}{t^2}\right)dt = e^t\left(\log t + \left(\frac{-1}{t}\right)\right) = x\left[\log\log x - \frac{1}{\log x}\right] + c.$$

95. $\int\sqrt{\frac{a^2}{x^2} - 1}\,\frac{dx}{x^3}$. [Substitute $\frac{a^2}{x^2} - 1 = t^2$]

96. $\int\frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\frac{\sin(x+a)}{\sin x}}} = \int\frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\cos a + \cot x \sin a}} = \frac{-1}{\sin a}\int\frac{dt}{\sqrt{t}}$ where $t = \cos a + \cot x \sin a$

97. $\int\frac{\sqrt{\cos 2x} \times \sin x}{\sin x \times \sin x}dx = \int\left(\frac{\sqrt{2\cos^2 x - 1}}{1 - \cos^2 x}\sin x\right)dx$ Put $\cos x = t$

$$\int\frac{\sqrt{2t^2 - 1}(-dt)}{1 + t^2} = \int\frac{2t^2 - 1}{(t^2 - 1)\sqrt{2t^2 - 1}}dt = 2\int\frac{1}{\sqrt{2t^2 - 1}}dt + \int\frac{dt}{(t^2 - 1)\sqrt{2t^2 - 1}}$$

98. $I = \int\frac{dx}{\left[a + b\left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)\right]^2} = \int\frac{\sec^4\frac{x}{2}dx}{\left[(a+b) + (a-b)\tan^2\frac{x}{2}\right]^2}$. [Substitute $(a-b)\tan^2\frac{x}{2} = (a+b)\tan^2\theta$]

$$\Rightarrow \quad \sqrt{(a-b)}\sec^2\frac{x}{2}dx = 2\sqrt{(a+b)}\sec^2\theta\, d\theta$$

$$\Rightarrow \quad I = \int\frac{\left(1 + \tan^2\frac{x}{2}\right)\sec^2\frac{x}{2}dx}{\left[a + b + (a-b)\tan^2\frac{x}{2}\right]^2} = 2\int\frac{\left[1 + \frac{a+b}{a-b}\tan^2\theta\right]\sqrt{\frac{a+b}{a-b}}\sec^2\theta\, d\theta}{(a+b)^2\sec^4\theta}$$

$$\Rightarrow \quad I = \frac{2}{(a^2 - b^2)^{3/2}}\int[(a-b)\cos^2\theta + (a+b)\sin^2\theta]d\theta$$

$$\Rightarrow \quad I = \frac{2}{(a^2 - b^2)^{3/2}}\left[\frac{a-b}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + \frac{a+b}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right] + c$$

$$\Rightarrow \quad I = \frac{2}{(a^2 - b^2)^{3/2}}\left[a\theta - \frac{b}{2}\sin 2\theta\right] = \frac{2}{(a^2 - b^2)^{3/2}}\left[a\theta - \frac{b\tan\theta}{1 + \tan^2\theta}\right] + c$$

Substitute $\tan^2\theta = \frac{(a-b)}{(a+b)}\tan^2\frac{x}{2}$ and simplify.

99. Write $x^3 - x + 2 = (x^2 + 1)(x + 1) - x^2 - 2x + 1 \Rightarrow I = \int e^x\left[\underbrace{\frac{(x^2 + 1)(x + 1)}{(x^2 + 1)^2}}_{f(x)} + \underbrace{\frac{1 - x^2 - 2x}{(x^2 + 1)^2}}_{f'(x)}\right]dx = \frac{e^x(x+1)}{x^2 + 1} + c$

100. Apply By parts taking $\sin^{-1}\left[\frac{1}{2}\sqrt{\frac{2a - x}{a}}\right]$ as I[st] part and x as II[nd] part.